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THE DISTANCE-REGULAR GRAPHS WITH INTERSECTION NUMBER $a_1 \neq 0$ AND WITH AN EIGENVALUE $-1-(b_1/2)$

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In this paper we will classify the distance-regular graphs with intersection number $a_1 \neq 0$ and with an eigenvalue $-(b_1/2)-1$.

1. Introduction

Terwilliger [8] showed that if $a_1 \neq 0$, then $\theta \geq -(b_1/2)-1$ for all eigenvalues θ of a distance-regular graph Γ, cf. [1, Theorem 4.4.3]. In this paper we will classify the distance-regular graphs with an eigenvalue $-(b_1/2)-1$ and $a_1 \neq 0$. The main result is the following theorem.

Theorem 1. *Let Γ be a distance-regular graph with intersection number $a_1 \geq 2$. Then Γ has eigenvalue $-b_1/2-1$ if and only if one of the following holds:*

(i) *Γ is a clique.*

(ii) *Γ is a connected strongly regular graph with second largest eigenvalue 1.*

(iii) *Γ is the distance-regular graph with intersection array $\{10,6,4,1;1,2,6,10\}$.*

In case of $a_1=1$ we will give the following combinatorial characterisation.

Theorem 2. *Let Γ be a distance-regular graph with $a_1=1$ and diameter d. Then Γ has an eigenvalue $-1-(b_1/2)$ if and only if there exist a $1 \leq j \leq d$ such that $a_i=c_i$ for $i<j$, $a_j=b_j+c_j$ and $a_i=b_i$ for $i>j$.*

Remarks. (i) The graphs in Theorem 1 (ii) are the complements of strongly regular graphs with smallest eigenvalue at least -2. These are classified by Seidel [6].

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(ii) The graph with intersection array $\{10,6,4,1;1,2,6,10\}$ is unique and locally the Petersen graph as shown by J. I. Hall [4].

In Section 2 we give some preliminary results, in Section 3 we classify the distance-regular graphs without an induced quadrangle and such that the local graphs have second largest eigenvalue at most 1, in Section 4 we look at distance-regular graphs which are locally the complement of a line graph, in Section 5 we give the proof of Theorem 1 and in Section 6 we give the proof of Theorem 2.

2. Preliminaries

All necessary definitions can be found in Brouwer, Cohen and Neumaier [1], and in Godsil [3]. For a graph Γ and a vertex $x \in V\Gamma$ denote the set $\{y \in V\Gamma \mid d(x,y) = i\}$ by $\Gamma_i(x)$. Instead of $\Gamma_1(x)$ we write $\Gamma(x)$. A *distance-regular* graph Γ is a connected graph with constants a_j, b_j, and c_j, $j = 1,2,\ldots,d$, where d is the diameter of Γ, such that the following hold for any x and y at distance i, $|\Gamma_{i-1}(x) \cap \Gamma(y)| = c_i$, $|\Gamma_i(x) \cap \Gamma(y)| = a_i$, and $|\Gamma_{i+1}(x) \cap \Gamma(y)| = b_i$. A *strongly regular* graph is a graph with constants λ and μ, such that the number $|\Gamma(x) \cap \Gamma(y)|$ only depends on whether x and y are adjacent or not. In the first case this number is denoted by λ, in the second case by μ. A connected non-complete strongly regular graph is a distance-regular graph of diameter 2. Let Γ be a distance-regular graph. By θ_1, we denote the second largest eigenvalue of Γ and by θ_d the smallest. For a vertex x of Γ we will denote the *local graph* on x, i.e. the graph induced by $\Gamma(x)$, by $\Delta(x)$. The starting point of this paper is the following proposition.

Proposition 3. *Let Γ be a distance-regular graph with an eigenvalue $-b_1/2-1$. Then the local graph $\Delta(x)$ is a graph with second largest eigenvalue at most 1.*

Proof. The local graph has second largest eigenvalue at most $-1-b_1/(\theta_d+1)$, cf. [1, Theorem 4.4.3]. Now the proposition follows easily, since $\theta_d = -b_1/2-1$. ∎

The following proposition shows that the local graphs are connected and co-connected. We write $\overline{\Gamma}$ for the complement of a graph Γ.

Proposition 4. *Let Γ be a distance-regular graph such that the local graphs $\Delta(x)$ have second largest eigenvalue at most 1 for all vertices x of Γ. Then the following hold:*

(i) *If $a_1 > 1$, then $\Delta(x)$ is connected.*

(ii) *If the diameter d is at least three then the complement of $\Delta(x)$, $\overline{\Delta(x)}$, is connected.*

Proof. (i): Trivial. (ii): Suppose that $\overline{\Delta(x)}$ is disconnected and $d \geq 2$. Then each component of $\overline{\Delta(x)}$ has at least b_1+1 vertices. Hence $c_2 \geq b_1+2$, by $d \geq 2$, and so $d = 2$. ∎

The complement of a graph with second largest eigenvalue 1 is a graph with smallest eigenvalue -2. We quote Theorem 5 without proof, for use in the proof of Theorem 1.

Theorem 5. (Bussemaker and Neumaier [2], cf. [1, Theorem 3.12.2]) *Let* Γ *be a connected regular graph with* v *vertices, valency* k *and smallest eigenvalue* ≥ -2. *Then one of the following holds:*

(i) Γ *is the line graph of a regular or a bipartite semi-regular graph.*

(ii) $v=2(k+2)\leq 28$.

(iii) $v=3(k+2)/2\leq 27$.

(iv) $v=4(k+2)/3\leq 16$.

(v) $v=k+2$ *and* $\Gamma=K_{m\times 2}$ *for some* $m\geq 3$. ∎

3. Graphs without induced quadrangles

In this section we classify the distance-regular graphs without an induced quadrangle, i.e. an induced 4-gon, and such that all local graphs have second largest eigenvalue at most 1. First we give some definitions.

Let Γ be a graph. Write for a vertex x of Γ, $x^{\perp}=\{x\}\cup\Gamma(x)$, and define an equivalence relation $\equiv$ on $V\Gamma$ by letting $x\equiv y$ if and only if $x^{\perp}=y^{\perp}$. We shall write $\tilde{\Gamma}$ for the quotient $\Gamma/\equiv$ and $\tilde{x}$ for the equivalence class of x. (I.e., $\tilde{\Gamma}$ has vertices $\tilde{x}$ for $x\in V\Gamma$ and $\tilde{x}\sim\tilde{y}$ when $x\sim y$ and $\tilde{x}\neq\tilde{y}$.) $\tilde{\Gamma}$ is called the *reduced graph* of Γ.

A *Terwilliger* graph is a non-complete graph Γ such that the graph induced by $\Gamma(x)\cap\Gamma(y)$ is a clique of size μ, $\mu\geq 2$, for any two vertices x and y at distance 2.

Note that a Terwilliger distance-regular graph is just a non-complete distance-regular graph without induced quadrangles.

We quote Theorems 6, 7 and 8 without proofs.

Theorem 6. (cf. [1, Theorem 3.12.4]) *Let* Γ *be a connected strongly regular non-complete graph with smallest eigenvalue* ≥ -1. *Then* Γ *is the pentagon, a triangular graph* $T(n)$, $(n\geq 3)$, *a square grid* $n\times n$, $(n\geq 3)$, *a complete multipartite graph* $K_{n\times 2}$, $(n\geq 2)$, *or one of the graphs of Petersen, Schläfli, Clebsch, Shrikhande, or Chang.* ∎

Theorem 7. (cf. [1, Theorem 1.16.5]) *There are up to isomorphism three connected locally Petersen graphs, namely:*

(i) *The complement of the triangular graph* $T(7)$.

(ii) *The distance-regular graph with intersection array* $\{10,6,4,1;1,2,6,10\}$.

(iii) *The distance-regular graph with intersection array* $\{10,6,4;1,2,5\}$. ∎

Theorem 8. (cf. [1, Theorem 1.16.3]) *Let* Γ *be a distance-regular Terwilliger graph with* $c_2 \geq 2$, *Fix* $x \in V\Gamma$. *Let* $\Delta(x)$ *be the subgraph induced by* $\Gamma(x)$. *Then the reduced graph* $\widetilde{\Delta(x)}$ *is strongly regular.* ∎

Proposition 9. *If* Γ *is a distance-regular Terwilliger graph with* $c_2 \geq 2$, *such that all the local graphs* $\Delta(x)$ *have second largest eigenvalue at most 1, then* Γ *is*

(i) *an icosahedron,*

(ii) *the distance-regular graph with intersection array* $\{10,6,4,1;1,2,6,10\}$, *or*

(iii) *the distance-regular graph with intersection array* $\{10,6,4;1,2,5\}$.

Proof. For all $x \in V\Gamma$ the local graph $\Delta(x)$ is connected, otherwise there would be induced quadrangles. Hence we find $a_1 \geq 2$. By Theorem 8 we find that the graph $\widetilde{\Delta(x)}$ must be a strongly regular graph with second largest eigenvalue at most 1. The complement of a graph with second largest eigenvalue at most 1 is a graph with smallest eigenvalue at least -2. By looking at the possibilities of Theorem 6, the graph $\widetilde{\Delta(x)}$ is a pentagon or the Petersen graph. In both cases we find $\Delta(x) = \widetilde{\Delta(x)}$. If Γ is locally the pentagon then Γ is the icosahedron. If Γ is locally the Petersen graph then by Theorem 7, Γ must be one of the last two graphs in the statement. ∎

4. Locally co-line graphs

The following inequalities of Terwilliger [7] will be used to bound the diameter of a distance-regular graph with an induced quadrangle.

Theorem 10. (Terwilliger [7], cf. [1, Theorem 5.2.1]) *Let* Γ *be a distance-regular graph with an induced quadrangle. Then*

$$c_i - b_i \geq c_{i-1} - b_{i-1} + a_1 + 2, \qquad (i = 1, 2, \ldots, d).$$ ∎

Terwilliger [9] classified the distance-regular graphs which have equality for $i = 1, 2, \ldots, d$ in the above inequalities.

Theorem 11. (Terwilliger [9], cf. [1, Theorem 5.2.3]) *Let* Γ *be a distance-regular graph with diameter* $d \geq (k + c_d)/(a_1 + 2)$. *Then one of the following holds:*

(i) Γ *does not have an induced quadrangle.*

(ii) Γ *is strongly regular with smallest eigenvalue* -2.

(iii) Γ *is a Hamming graph, a Doob graph, a locally Petersen graph, a Johnson graph, a halved cube, or the Gosset graph.* ∎

Proposition 12. *Let* Γ *be a distance-regular graph with diameter* $d \geq 3$, $a_1 \geq 2$ *and an eigenvalue* $-1 - (b_1/2)$, *which has an induced quadrangle and such that for all vertices* x *the graph* $\Delta(x)$ *induced on* $\Gamma(x)$ *is connected and the complement of a* *line graph. Then the pair* (k,a_1) *is one of the following:* $(27,16)$, $(24,13)$, $(20,11)$, $(15,6)$, $(14,7)$, $(12,5)$, $(10,3)$, *or* $(9,4)$.

Proof. We may assume that $d\geq 3$. Let x be a vertex of Γ. The graph $\Delta(x)$ is the complement of a line graph of a graph, say $\Lambda(x)$. The graphs $\Delta(x)$ and $\Lambda(x)$ are connected by Proposition 4 and by Theorem 5 we have two cases: $\Lambda(x)$ is bipartite semi-regular with two different valencies, or $\Lambda(x)$ is regular.

Case 1. $\Lambda(x)$ is semi-regular with two different valencies.

Let $l<m$ be the two valencies of $\Lambda(x)$. Then $k\geq lm$ and $b_1=l+m-2$. By Theorem 10 it follows that $k\leq 3b_1-3$ and hence $lm\leq 3l+3m-9$. If $l\geq 4$, then $m\leq 3$, contradiction. So $l\leq 3$.

If $l=3$, then $d\leq 3$, $k=3m$ and $b_1=m+1$. The case $d=3$ is excluded by Theorem 11, as $k=3m$ and $b_1=m+1$.

If $l=2$, then $k\leq 3m$–3. If $k>2m$, then $k\geq 3m$. Hence $k=2m$ and $\Lambda(x)=K_{2,m}$. It follows that $c_2\geq m-1$. Now $k_2=kb_1/c_2=2m^2/c_2$, and it follows that $c_2\geq m$, unless we are in the case $m=3$. So Γ is a Taylor graph, cf. [1, Theorem 1.5.5], or $m=3$. A Taylor graph is locally a strongly regular graph, cf. [1, Theorem 1.5.3] and the line graph of $K_{2,m}$ is only strongly regular if $m\leq 2$, and hence we have $m=3$ or $d\leq 2$.

If $l=1$, then $b_1=m-1$ and $c_2\geq m$ when $k\geq m+1$. So $d\leq 2$.

Case 2. $\Lambda(x)$ is regular.

We may assume that $\Lambda(y)$ is regular for all $y\in V\Gamma$. We denote by w the number of vertices of $\Lambda(x)$, and by l the valency of $\Lambda(x)$. Then $k=wl/2$ and $b_1=2(l-1)$. By Theorem 10 it follows that $k\leq 3b_1-3$, and hence $wl\leq 12l-18$. If $w=l+1$, then $\Lambda(y)$ is a clique of size $l+1$ for all vertices y and J.I. Hall showed in [5] that $d=2$ if $w\geq 7$. So we may assume that $w\geq l+2$. Then we obtain $l\leq 7$. Now by checking all pairs (w,l) with $l\leq 7$, satifying $wl\leq 12l-18$ and wl is even, we find the pairs in the proposition. ∎

5. Proof of Theorem 1

Let Γ be a distance-regular graph with eigenvalue $-b_1/2-1$. If $d=1$, then Γ is complete and we are in case (i). If $d=2$, then Γ is strongly regular and the eigenvalues $\theta_1=r$ and $\theta_2=s$ satisfy the equation $x^2+(c_2-a_1)x+c_2-k=0$. Therefore, $r+s=a_1-c_2$, $rs=c_2-k$ and by the hypothesis, $b_1(r+1)=-2(r+1)(s+1)=2k-2a_1-2=2b_1$. Since $b_1\neq 0$ we find $r=1$, and (ii) holds.

By Proposition 3, all the subgraphs $\Delta(x)$ have second largest eigenvalue at most 1. Suppose first that Γ does not contain an induced quadrangle. Then, by Proposition 9, Γ is the icosahedron or Γ has intersection array $\{10,6,4,1;1,2,6,10\}$ or $\{10,6,4;1,2,5\}$. But the icosahedron and the distance-regular graph with intersection array $\{10,6,4;1,2,5\}$ do not have an eigenvalue $-b_1/2-1$.

Otherwise Γ has an induced quadrangle. The complement of a graph with second largest eigenvalue at most 1 has smallest eigenvalue at least -2. By Proposition 4, the complement of the local graph $\Delta(x)$, $\overline{\Delta(x)}$ is connected, hence we can apply Theorem 5. If $\overline{\Delta(x)}$ is not a line graph then the pair (k,b_1) is one of the following, $(2(t+2),t)$, $3\le t\le 12$, $(3(u+1),2u)$, $1\le u\le 8$, $(4v,3v-2)$, $2\le v\le 4$. If $\overline{\Delta(x)}$ is a line graph then, by Proposition 12, the pair (k,b_1) is one of the following, $(27,10)$, $(24,10)$, $(20,8)$, $(15,8)$, $(14,6)$, $(12,6)$, $(10,6)$, $(9,4)$. Using Theorem 10, we deduce that $d\le 4$ and $|V\Gamma|\le 900$. (This is obtained by looking at the pair $(27,10)$.) Therefore the intersection array of Γ is included in the tables of [1, Chapter 14]. By checking each possible pair (k,b_1) we find that Γ is either the unique graph with intersection array $\{10,6,4;1,2,5\}$, or an antipodal graph of diameter 3. In the first case, the graph has no quadrangles, a contradiction. In the second case, for each pair (k,b_1) we calculate the two non-trivial eigenvalues using the information of [1, page 431], assuming that $\theta_d=-(b_1/2)-1$. But then in none of the cases we find that c_2 divides b_1, which should be the case. So we get a contradiction.

It is easy to see that in all the three cases of Theorem 1 the graphs have an eigenvalue $-(b_1/2)-1$. So this completes the proof of the theorem. ∎

6. Proof of Theorem 2

In this section we will give a combinatorial characterisation of the distance-regular graphs with $a_1=1$ and an eigenvalue $-b_1/2-1$. First we need some definitions. Let Γ be a graph. Let $\emptyset\neq C\subset V\Gamma$. Define for a vertex x of Γ the distance $d(x,C)$ as the minimum of $d(x,y)$ where $y\in C$. Define for a subset $\emptyset\neq C\subseteq V\Gamma$ and an integer i, the set C_i by $C_i:=\{y\in V\Gamma\,|\,d(y,C)=i\}$. Furthermore define $\rho(C)$ as the maximum i such that $C_i\neq\emptyset$. A subset $\emptyset\neq C\subset V\Gamma$ is called a *completely regular code* if the cardinality of $\Gamma_j(x)\cap C$ only depends on j and $d(x,C)$ The following lemma gives a bound on the size of a complete subgraph in a distance regular graph and is due to Godsil.

Lemma 13. (cf. [3, Chapter 13, Lemmata 7.1, 7.2]) *Let Γ be a distance-regular graph with valency k, diameter d and smallest eigenvalue θ_d. If C is a clique of Γ, then*

$$|C|\le 1+\frac{k}{-\theta_d}.$$

If equality holds then C is a completely regular code with $\rho(C)=d-1$ and its parameters are uniquely determined.

Proof. Let E be the primitive idempotent with respect to θ_d. Define for a vertex x the vector $\hat{x}$ by $\hat{x}_y:=\delta_{xy}$ for $y\in V\Gamma$, where δ_{xy} denotes the Kronecker delta. Define $\hat{\gamma}$ by $\hat{\gamma}:=\sum_{x\in C}\hat{x}$, and for $i=0,1,\ldots,d$ define u_i by $u_i:=\hat{y}^TE\hat{x}$, where x,y are two vertices at distance i. Define the matrix M by

$$M := \hat{\gamma}^T E \hat{\gamma}.$$

We have $M = u_1 J + (1 - u_1) I$. Note that $u_1 = \theta_d / k$. The matrix M is positive semi-definite and therefore $1 + (|C| - 1) u_1 \geq 0$. The inequality follows easily.

Furthermore equality holds if and only if $1 + (|C| - 1) u_1 = 0$ and therefore if and only if $E\hat{\gamma} = \mathbf{0}$. Let y be at distance i from C. Let $\alpha(y)$ be the cardinality of $\Gamma_i(y) \cap C$. Then $0 = (E\hat{y})^T E\hat{\gamma} = \alpha(y) u_i + (|C| - \alpha) u_{i+1}$. This means that $\alpha(y)$ does not depend on y but only on the distance $d(y, C)$, otherwise $u_i = u_{i+1} = 0$, which is impossible for a distance-regular graph. It also follows that $d(y, C) \leq d - 1$, otherwise $u_d = 0$, again an impossibility for a distance-regular graph. So we are done. ∎

Proof of Theorem 2. The if part is not difficult to see.

By Lemma 13 we find that a triangle C in Γ is a completely regular code with covering radius $\rho(C) = d - 1$. Let x be a vertex. Let y be a vertex of Γ at distance $i \geq 1$ from x. Now let C be a triangle of Γ such that $y \in C$ and $d(x, C) = i - 1$. Because by Lemma reflemgod the parameters of C as completely regular code are uniquely determined, the theorem follows now easily. ∎

Remarks. (i) Of course, for bipartite graphs the smallest eigenvalue is $-k$ and this is usually smaller then $-1 - (b_1/2)$. Also for $a_1 = 0$ there are many examples known where $-1 - (b_1/2)$ is an eigenvalue.

(ii) For $a_1 = 1$ all the regular near $2n$-gons have $-1 - (b_1/2)$ as an eigenvalue and there are many examples known. Except for those regular near $2n$-gons the author knows of only one example with $a_1 = 1$ and eigenvalue $-1 - (b_1/2)$, namely the distance-regular graph with intersection array $\{6, 4, 2, 1; 1, 1, 4, 6\}$.

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